$$Y_\eta'(0,\ p) = -\left(\frac{1}{2}\Pr f''(0)\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'}{\mathrm{Ai}}\left[\left(\frac{2}{\Pr f''(0)}\right)^{\frac{2}{3}} p\right]. \tag{A4}$$

It is interesting to note that by setting p = 0 in (A4) we obtain

$$Y_\eta'(0) \approx -\left(\frac{1}{2}\Pr f''(0)\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = \frac{3}{\Gamma\left(\frac{1}{3}\right)}\left(\frac{\Pr f''(0)}{6}\right)^{\frac{1}{3}}.$$

This result agrees well with the first term of the expansion (A2), according to which

$$Y_0'(0) = \frac{3}{(1/3)}\left(\frac{\Pr f''(0)}{6}\right)^{1/3}.$$

Thus, for moderate Pr ($1 \leq \Pr \ll \infty$) Eq. (A4) can be used for all p.

As $p \to \infty$ ($|p| \gg [{}^1/{}_2 \Pr f''(0)]^{2/3}$) we find $Y_\eta'(0, p) \approx \sqrt{p}$, while for $\eta \ll 2|p|/\Pr f''(0)$ we have $Y(\eta,\ p) \approx \exp(-\sqrt{p}\eta)$. These equations correspond to the case of pure heat conduction.

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HEAT PROPAGATION BY HEAT CONDUCTION IN ACTIVE LINEAR MEDIA

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A method to use the matrix A-parameter method [1] to solve linear heat-conduction problems in active media is proposed.

The system of differential equations describing the temperature and heat flux distribution in an inhomogeneous heat line (IHL) within which distributed heat and temperature sources act has the form [1]

$$\frac{\partial t}{\partial r} = -R_l q - I_l \frac{\partial q}{\partial \tau} + E, \tag{1}$$

$$\frac{\partial q}{\partial r} = -g_l t - c_l \frac{\partial t}{\partial \tau} + P. \tag{2}$$

Equations (1)-(2) form a system of so-called telegraph equations in which the effect of the internal distributed sources is taken into account. The case when the distributed temperature sources (E) and the distributed heat sources (P) are independent, i.e., are dependent on neither the temperature nor the heat flux, but at the same time can be given as functions of the coordinates or time, has been examined earlier [1]. It is shown there how a problem with given initial conditions reduces to a problem with independent heat sources. In this paper the case when the distributed sources of both E and P depend linearly on the temperature or on the heat flux (or on their time rate of change) is examined.

Let us consider the following variants:

1a) $E = R_{l+}q(r,\ \tau)$ are the distributed temperature sources proportional to the heat flux;

1b) $E = I_{l+}\partial q(r,\ \tau)/\partial\tau$ are the distributed temperature sources proportional to the time

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rate of change of the heat flux;

2) $E = k_{l+}t(r, \tau)$ are the distributed temperature sources proportional to the temperature;

3a) $P = g_{l+}t(r, \tau)$ are the distributed heat sources proportional to the temperature;

3b) $P = c_{l+}\partial t(r, \tau)/\partial r$ are the distributed heat sources proportional to the time rate of change of the temperature;

4) $P = l_{l+}q(r, \tau)$ are the distributed heat sources proportional to the heat flux.

Here $R_{l+}I_{l+}k_{l+}$, $g_{l+}c_{l+}L_{l+}$ are proportionality factors. We shall henceforth speak of the distributed sources of the type (1-3) and (2-4) and, for brevity, of the heat lines (1-3) and (2-4).

Let us consider the heat lines (1-3). In this case the system of linear differential equations (1) and (2) can be represented in the expanded form

$$\frac{\partial t}{\partial r} = -R_l q + R_{l+} q - I_l \frac{\partial q}{\partial \tau} + I_{l+} \frac{\partial q}{\partial \tau} = -(R_l - R_{l+})q - (I_l - I_{l+}) \frac{\partial q}{\partial \tau} = -R_l^* q - I_l^* \frac{\partial q}{\partial \tau}, \tag{3}$$

$$\frac{\partial q}{\partial r} = -g_l t + g_{l+} t - c_l \frac{\partial t}{\partial \tau} + c_{l+} \frac{\partial t}{\partial \tau} = -(g_l - g_{l+})t - (c_l - c_{l+}) \frac{\partial t}{\partial \tau} = -g_l^* t - c_l^* \frac{\partial t}{\partial \tau}, \tag{4}$$

where $R_l^* = R_l - R_{l+}$; $I_l^* = I_l - I_{l+}$; $g_l^* = g_l - g_{l+}$; $c_l^* = c_l - c_{l+}$.

When no sources but sinks, act in the body, the plus sign in front of the coefficients with the subscript + is replaced by a minus. The coefficients marked with an asterisk have the meaning of effective parameters. As is seen from (3) and (4), taking account of the action of the dependent sources of type (1-3) results only in a diminution of the differential-equation coefficients up to obtaining negative values. Therefore, the usual methods used in the analysis of passive heat lines [1] are applicable to the computation of active heat lines of the type (1-3). For both passive and active heat lines (1-3), inhomogeneous in the general case, the reciprocity principle in the coordinates is satisfied.

A thermal system with the sources (1-3) can turn out to be unstable. Thus, let us examine the system function $y_{11} = \bar{q}_1/\bar{t}_1|_{t_2=0}$ for a homogeneous heat line (HHL) (1-3). Let the physical parameters of the body be $R_{l_0}^*$, c_{l_0}, $g_{l_0}^*$. The singularity of the active HHL (1-3) under consideration is that both $g_{l_0}^*$ and $R_{l_0}^*$ can have a negative sign. As follows from an analysis of the system function y_{11}, a change in the sign of the parameters $g_{l_0}^*$ to the negative results in a shift of its zeroes and poles to the right, and for the $R_{l_0}^*$ to inversion of its zeroes and poles from the negative to the positive half plane. The system will be stable in the mode under consideration upon compliance with the conditions $R_{l_0}^* > 0$ and $g_{l_0}^* > 0$ or $R_{l_0}^* > 0$ and $g_{l_0}^* < 0$, but $|g_{l_0}^*| < \pi^2/r^2 R_{l_0}^*$.

The stability of other problems for both the HHL and the IHL (1-3) can be investigated in an analogous manner.

Heat Lines (2-4). In this case the system of two partial differential equations describing the process of heat transfer by heat conduction will have the form

$$\frac{\partial t}{\partial r} = (-R_l(r)\, q - I_l(r) \frac{\partial q}{\partial \tau}) + k_{l+}(r) t, \tag{5}$$

$$\frac{\partial q}{\partial r} (-g_l(r) t - c_l(r) \frac{\partial t}{\partial \tau}) + l_{l+}(r) q. \tag{6}$$

After executing a Laplace transformation of the equations presented above with zero initial conditions taken into account, we have

$$\frac{dt}{dr} = -z_l(r)\bar{q} + k_{l+}(r)\,\bar{t}, \tag{7}$$

$$\frac{d\bar{q}}{dr} = -y_l(r)\bar{t} + l_{l+}(r)\bar{q}, \tag{8}$$

where $z_l(r) = R_l(r) + sI_l(r)$; $y_l(r) = g_l(r) + sc_l(r)$.

For a homogeneous heat line (HHL) [$\sigma(r)$ = const and the coefficients z_{l0}, y_{l0}, k_{l0+}, l_{l0+} are independent of the coordinates and are constants] the following heat conduction equation in Laplace transforms is derived on the basis of (5) and (6):

$$\frac{d^2\bar{t}}{dr^2} - (k_{l0+}+l_{l0+})\frac{d\bar{t}}{dr} - (z_{l0}y_{l0} - k_{l0+}l_{l0+})\bar{t} = 0. \tag{9}$$

The differential equation in Laplace transforms for the initial-heat-conduction function $y_{in} = \bar{q}(r, s)/\bar{t}(r, s)$, obtained on the basis of (7) and (8), has the form

$$y_l(r) + \frac{dU_{in}}{dr} = z_l U_{in}^2 - (k_{l0+}(r) - l_{l0+}(r))U_{in}. \tag{10}$$

Let us examine the method to obtain the A-parameter matrix [1] for the inhomogeneous heat line (IHL) (2-4). It is known that a differential equation formed for the A-parameter matrix has the form

$$\frac{d}{dr}[A(r)] = -[A(r)][X(r)], \tag{11}$$

where the matrix $[X(r)]$ can be composed on the basis of the system of differential equations (7) and (8) written in the matrix form

$$\frac{d}{dr}\begin{bmatrix}\bar{t}\\ \bar{q}\end{bmatrix} = [X(r)]\begin{bmatrix}\bar{t}\\ \bar{q}\end{bmatrix} = \begin{bmatrix}k_{l+}(r) & -z_l(r)\\ -y_l(r) & l_{l+}(r)\end{bmatrix}\begin{bmatrix}\bar{t}\\ \bar{q}\end{bmatrix}, \tag{12}$$

i.e.,

$$[X(r)] = \begin{bmatrix}X_{11}(r) & X_{12}(r)\\ X_{21}(r) & X_{22}(r)\end{bmatrix} = \begin{bmatrix}k_{l+}(r) & -z_l(r)\\ -y_l(r) & l_{l+}(r)\end{bmatrix}. \tag{13}$$

Solving the matrix equation (11) presented, the A-parameter matrix can be obtained in the general case for an inhomogeneous active heat line (2-4) in analytic form. In the IHL (2-4) case when the matrix elements $[x]$ are constants (independent of the coordinate r), i.e.,

$$[X] = \begin{bmatrix}k_{l0+} & -z_{l0}\\ -y_{l0} & l_{l0+}\end{bmatrix},$$

the solution of the matrix differential equation is written at once:

$$[A] = \exp\{[F]r\}, \tag{14}$$

where

$$[F] = -[X] = \begin{bmatrix}-k_{l0+} & z_{l0}\\ y_{l0} & -l_{l0+}\end{bmatrix}.$$

Therefore

$$[A] = \begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}\exp\left[-\frac{1}{2}(k_{l0+}+l_{l0+})r\right]\left[\mathrm{ch}(\gamma r) - \frac{1}{2}(k_{l0+}-l_{l0+})\frac{\mathrm{sh}(\gamma r)}{\gamma}\right] & z_{l0}\exp\left[-\frac{1}{2}(k_{l0+}+l_{l0+})r\right]\frac{\mathrm{sh}(\gamma r)}{\gamma}\\ y_{l0}\exp\left[-\frac{1}{2}(k_{l0+}+l_{l0+})r\right]\frac{\mathrm{sh}(\gamma r)}{\gamma} & \exp\left[-\frac{1}{2}(k_{l0+}+l_{l0+})r\right]\left[\mathrm{ch}(\gamma r) - \frac{1}{2}(l_{l0+}-k_{l0+})\frac{\mathrm{sh}(\gamma r)}{\gamma}\right]\end{bmatrix}, \tag{15}$$

where $\gamma = \sqrt{{}^1/_4(k_{l0+} - l_{l0+})^2 + z_{l0}y_{l0}}$, from which it follows that $AD - BC = e^{-(k_{l0+}+l_{l0+})r}$.

We represent the solution (11) as a power series in s:

$$[A] = \begin{bmatrix}A & B\\ C & D\end{bmatrix} = E - \int_0^r [X(r_1)]dr_1 + \int_0^r\left(\int_0^{r_1}[X(r_2)]dr_2\right)[X(r_1)]\,dr_1 - \int_0^r\left(\int_0^{r_1}\left(\int_0^{r_2}[X(r_3)]dr_3\right)[X(r_2)]\,dr_2\right)[X(r_1)]dr_1 + \ldots \tag{16}$$

On the basis of (16), for an active HHL (2-4)

$$A=1-k_{l0+}r+\frac{r^2}{2!}\left\{k^2_{l0+}+z_{l0}y_{l0}\right\}-\frac{r^3}{3!}\left\{k^3_{l0+}+z_{l0}y_{l0}(l_{l0+}+2k_{l0+})\right\}+\ldots, \tag{17}$$

$$B=z_{l0}r-\frac{r^2}{2!}\left\{z_{l0}(k_{l0+}+l_{l0+})\right\}+\frac{r^3}{3!}\left\{y_{l0}z^2_{l0}+z_{l0}(k^2_{l0+}+l_{l0+}k_{l0+}+l^2_{l0+})\right\}\ \ldots, \tag{18}$$

$$C=y_{l0}r-\frac{r^2}{2!}\left\{y_{l0}(k_{l0+}+l_{l0+})\right\}+\frac{r^3}{3!}\left\{z_{l0}y^2_{l0}+y_{l0}(k^2_{l0+}+l_{l0+}k_{l0+}+l^2_{l0+})\right\}-\ldots, \tag{19}$$

$$D=1-l_{l0+}r+\frac{r^2}{2!}\left\{l^2_{l0+}+z_{l0}y_{l0}\right\}-\frac{r^3}{3!}\left\{l^3_{l0+}+z_{l0}y_{l0}(k_{l0+}+2l_{l0+})\right\}+\ldots, \tag{20}$$

and for an active inhomogeneous heat line IHL (2-4)

$$A=1-\int_0^r k_{l+}(r_1)dr_1+\int_0^r\left\{\int_0^{r_1}k_{l+}(r_2)dr_2\right\}k_{l+}(r_1)dr_1+\int_0^r\left\{\int_0^{r_1}z_l(r_2)dr_2\right\}y_l(r_1)dr_1-\ldots, \tag{21}$$

$$B=\int_0^r z_l(r_1)dr_1-\int_0^r\left\{\int_0^{r_1}k_{l+}(r_2)dr_2\right\}z_l(r_1)dr_1-\int_0^r\left\{\int_0^{r_1}z_l(r_2)dr_2\right\}l_{l+}(r_1)dr_1+\ldots, \tag{22}$$

$$C=\int_0^r y_l(r_1)dr_1-\int_0^r\left\{\int_0^{r_1}y_l(r_2)dr_2\right\}k_{l+}(r_1)\,dr_1-\int_0^r\left\{\int_0^{r_1}l_{l+}(r_2)dr_2\right\}y_l(r_1)dr_1+\ldots, \tag{23}$$

$$D=1-\int_0^r l_{l+}(r_1)dr_1+\int_0^r\left\{\int_0^{r_1}y_l(r_2)dr_2\right\}z_l(r_1)dr_1+\int_0^r\left\{\int_0^{r_1}l_{l+}(r_2)dr_2\right\}l_{l+}(r_1)dr_1-\ldots. \tag{24}$$

The desired transfer function of the active thermal object [with the sources (2-4)] can be written on the basis of the A-parameter matrix for given boundary conditions, its stability can be determined, and the possible stable modes of a distributed gain in temperature or in heat flux can also be analyzed.

Heat lines with the sources (2-4) are irreversible in the coordinate. Any thermal four-pole is considered reversible in the coordinate if

$$\det[A]=AD-BC=1. \tag{25}$$

But from the Jacobi identity

$$1/(\det[A])\equiv\exp\left[\int_0^r \mathrm{Sp}[X(r)]dr=\exp\left[\int_0^r(X_{11}(r)+X_{22}(r))dr\right]\right], \tag{26}$$

it follows that det[A] = 1 if and only if $\int_0^r(X_{11}(r)+X_{22}(r))dr=0.$

Consequently, it can be stated that in the general case both the IHL and HHL will be irreversible in the coordinate if a distributed heat source of the type (2) and (or) (4) acts within them. Heat lines with sources (2-4) will be reversible in the coordinate only in the particular case when the condition $\int_0^r k_{l+}(r_1)dr_1+\int_0^r l_{l+}(r_1)dr_1=0$ is satisfied for the IHL (2-4), and $k_{l0+}+l_{l0+}=0$ for the HHL (2-4).

NOTATION

r, coordinate; τ, time; t, temperature; q, heat flux; σ(r), cross-sectional area in the heat propagation direction; R_l, linear thermal resistance; c_l, linear specific heat; g_l, linear heat conduction; and I_l, linear thermal inertia.

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